Source Positioning for Multi-Channel Gravitational Wave Data

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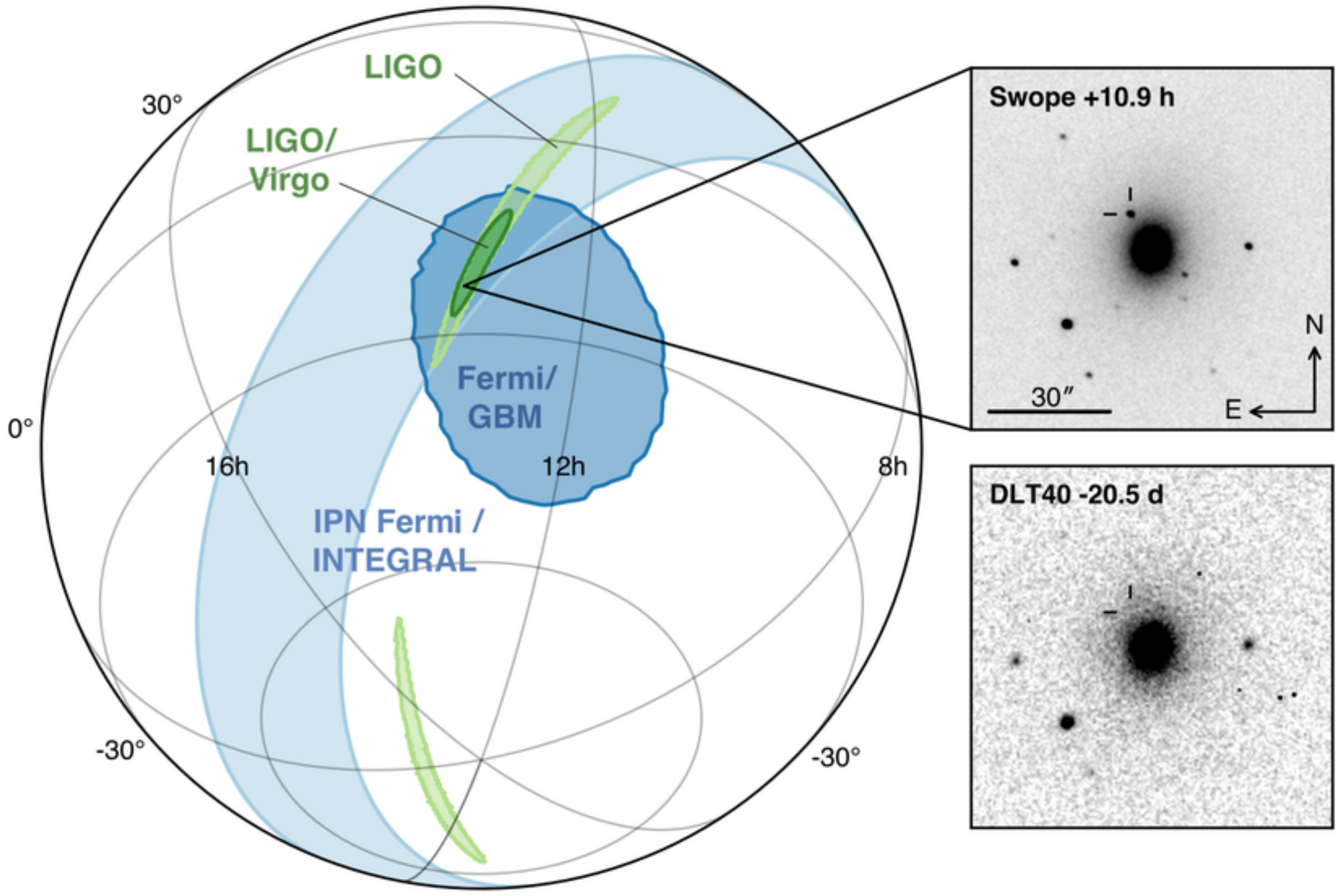
Motivations

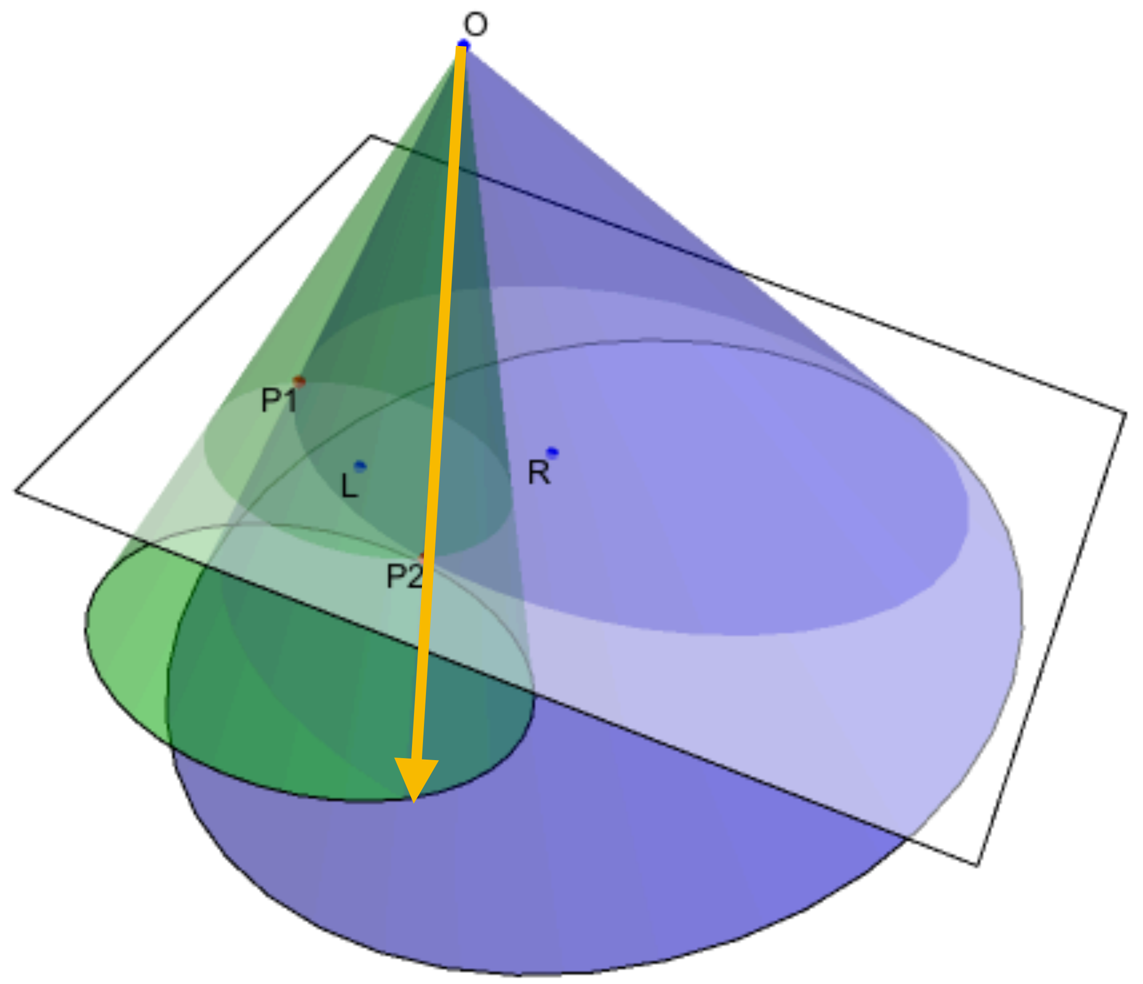
- Pointing to the source direction of gravitational waves (GWs) is essential for multi-messenger astronomy.
- The conventional method for determining the direction relies on phase differences among multiple detectors.
- However, multi-channel detectors such as SOGRO do not provide sufficient phase differences across their channels to identify the source direction.
- Instead of phase difference, SOGRO can exploit the tensorial components of the metric perturbation to determine the source direction, modulo 180 degree.
- We aim to establish the theoretical foundation for source localization in multi-channel detectors.



Sky localization of GW170817

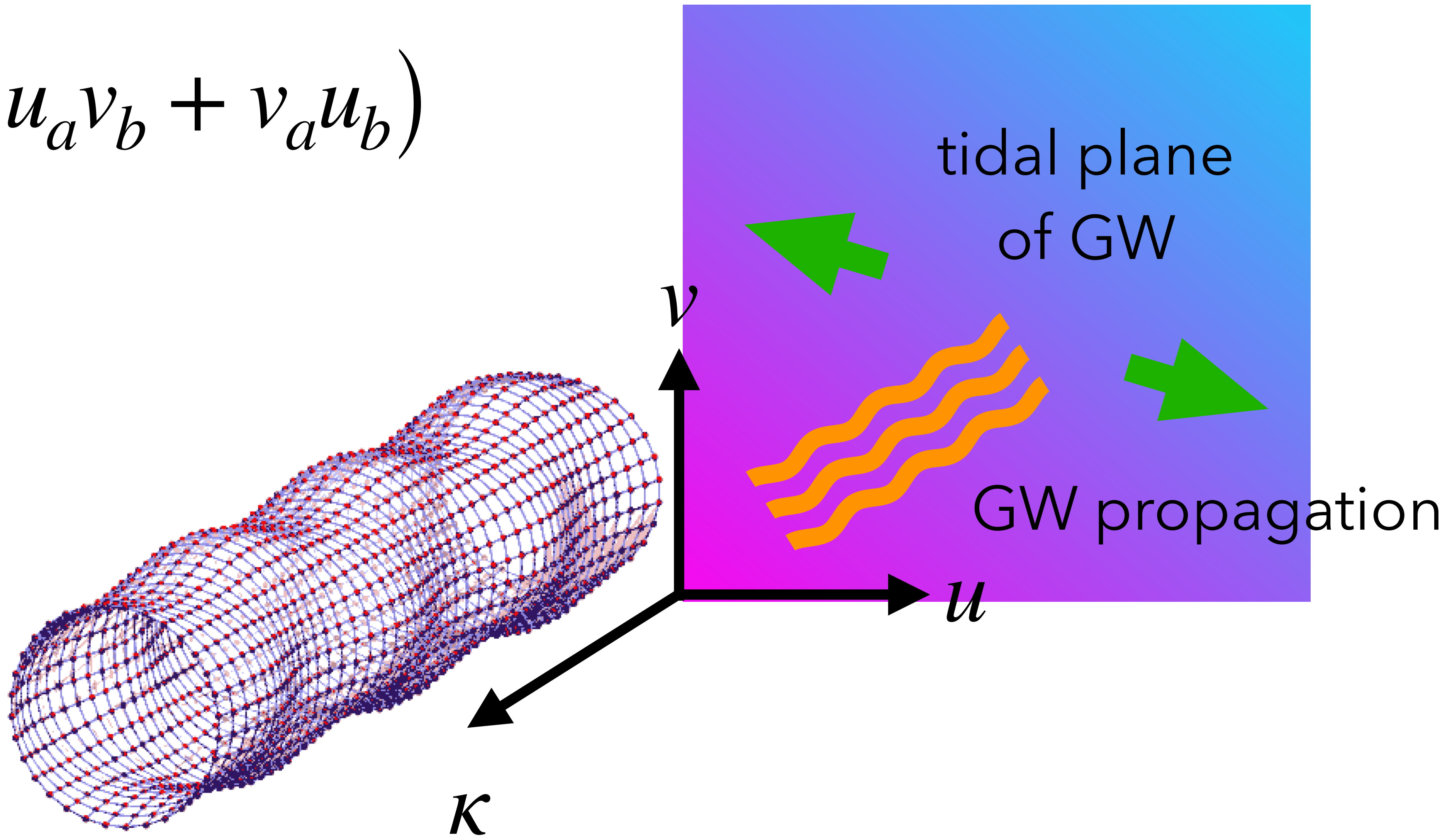
Conventional Method for Source Positioning

- Plane Waves
  - $\psi\left(t,\vec{x}\right) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\psi}(\omega)\, e^{i\omega\left(-t+\kappa\cdot\vec{x}\right)}$
- Measurements at different positions
  - $h_1(t) = \psi\left(t,\vec{x}_1\right) \quad \tilde{h}_1(\omega) = \tilde{\psi}(\omega)\, e^{i\omega\left(\kappa\cdot\vec{x}_1\right)}$
  - $h_2(t) = \psi\left(t,\vec{x}_2\right) \quad \tilde{h}_2(\omega) = \tilde{\psi}(\omega)\, e^{i\omega\left(\kappa\cdot\vec{x}_2\right)}$
  - $h_3(t) = \psi\left(t,\vec{x}_3\right) \quad \tilde{h}_3(\omega) = \tilde{\psi}(\omega)\, e^{i\omega\left(\kappa\cdot\vec{x}_3\right)}$
- Triangulation
  - $\left|\tilde{\psi}\right|^2 = \left|\tilde{h}_1\right|^2 = \left|\tilde{h}_2\right|^2 = \left|\tilde{h}_3\right|^2$
  - $\mathrm{Arg}\left(\tilde{h}_1^*\tilde{h}_2/\left|\tilde{\psi}\right|^2\right) = \omega\left(\kappa\cdot\left(\vec{x}_2-\vec{x}_1\right)\right)$
  - $\mathrm{Arg}\left(\tilde{h}_1^*\tilde{h}_3/\left|\tilde{\psi}\right|^2\right) = \omega\left(\kappa\cdot\left(\vec{x}_3-\vec{x}_1\right)\right)$



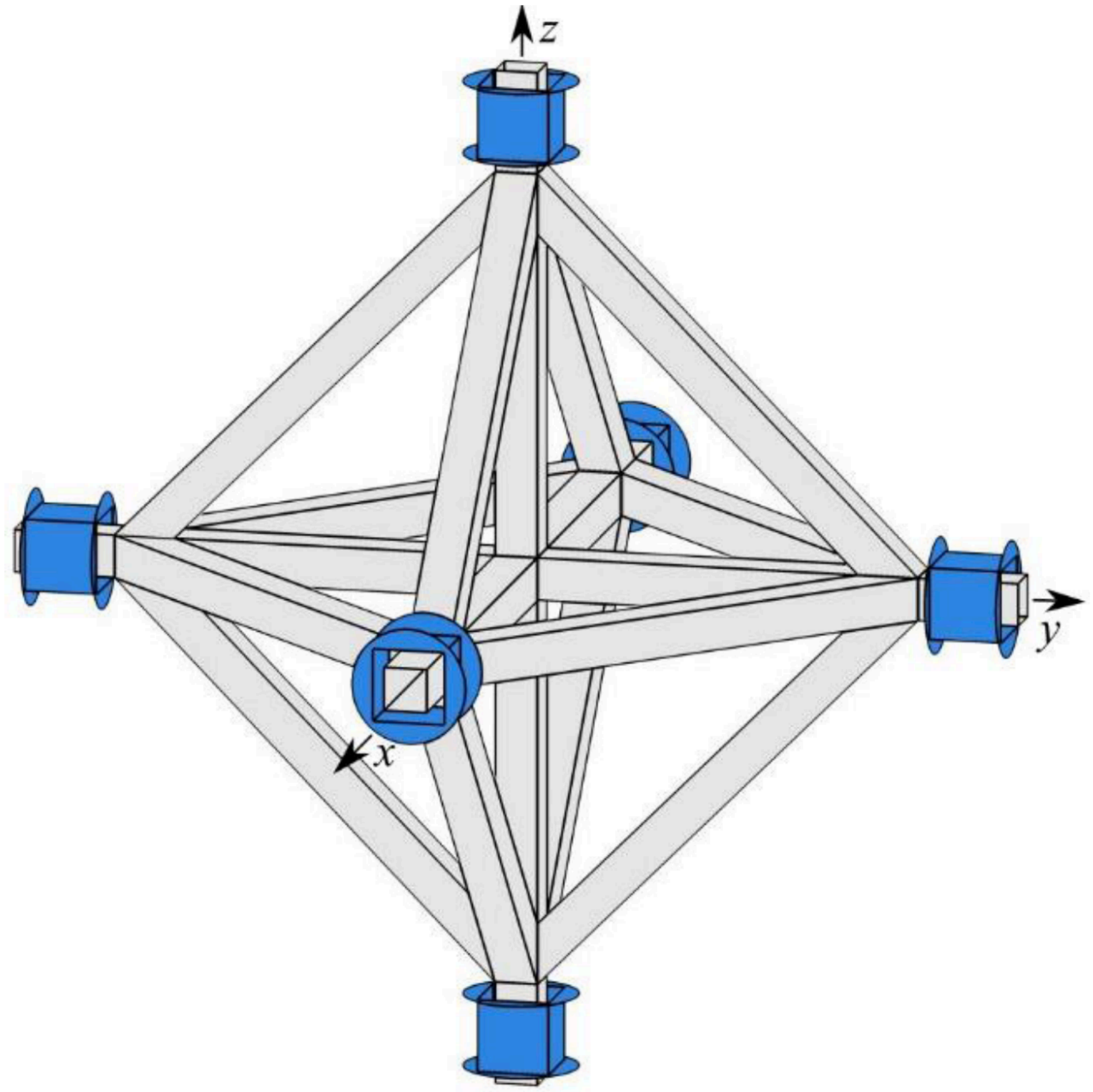
Plane Gravitational Waves (GWs)

- Metric perturbation in TT gauge for a plane GWs propagating to κ
  - $h_{ab}\left(t,\vec{x}\right) = h_{+}\left(t-\kappa\cdot\vec{x}\right) e^{+}_{ab}(\kappa) + h_{\times}\left(t-\kappa\cdot\vec{x}\right) e^{\times}_{ab}(\kappa)$
  - $\kappa^{a}\left(\theta,\phi\right) = \sin\theta\cos\phi\hat{x}^{a} + \sin\theta\sin\phi\hat{y}^{a} + \cos\theta\hat{z}^{a}$
- where
  - $e^{+}_{ab}(\kappa) = \frac{1}{\sqrt{2}}\left(u_a u_b - v_a v_b\right)$ $\qquad$ $e^{\times}_{ab}(\kappa) = \frac{1}{\sqrt{2}}\left(u_a v_b + v_a u_b\right)$
  - (u, v, κ): right-handed orthonormal basis
- such that
  - $e^{A}_{ab}e^{B}_{cd}g^{ac}g^{bd} = \delta^{AB}$ $\qquad$ for $A, B \in \{+, \times\}$



Detector Tensor and GW Signal

- D_i^{ab}: detector tensor for i-th channel
- $h_i(t) \equiv D_i^{ab} h_{ab}\left(t, \vec{x}_0\right)$: GW signal from i-th channel
- Examples
  - Interferometric detector (single channel)
    - $D^{ab} = \hat{x}^a \hat{x}^b - \hat{y}^a \hat{y}^b$ $\quad h = h_{xx} - h_{yy}$
  - SOGRO (multi-channel)
    - $D_1^{ab} = \hat{x}^a \hat{x}^b$ $\quad D_2^{ab} = \hat{y}^a \hat{y}^b$ $\quad D_3^{ab} = \hat{z}^a \hat{z}^b$
    - $D_4^{ab} = \frac{1}{2}\left(\hat{x}^a \hat{y}^b + \hat{y}^a \hat{x}^b\right)$ $\quad D_5^{ab} = \frac{1}{2}\left(\hat{y}^a \hat{z}^b + \hat{z}^a \hat{y}^b\right)$ $\quad D_6^{ab} = \frac{1}{2}\left(\hat{z}^a \hat{x}^b + \hat{x}^a \hat{z}^b\right)$
    - $h_1 = h_{xx}$ $\quad h_2 = h_{yy}$ $\quad h_3 = h_{zz}$
    - $h_4 = h_{xy}$ $\quad h_5 = h_{yx}$ $\quad h_4 = h_{zx}$



Source Positioning

- Given $h_{ab}\left(t,\vec{x}_0\right)$ at a position $\vec{x}_0$,
- Method 1. Use the explicit formula given in (Bae et al., PTEP 2024)
  - $\tan\theta = \dfrac{h_{11}+h_{22}}{h_{13}\cos\phi + h_{23}\sin\phi}$ $\qquad$ $\tan\left(2\phi\right) = 2\dfrac{h_{12}\left(h_{11}+h_{12}\right)+h_{13}h_{23}}{h_{11}^2 - h_{22}^2 + h_{13}^2 - h_{23}^2}$
- Method 2. Find an eigenvector of h_{ab} with zero eigenvalue. Equivalently, find a κ such that $h_{ab}\kappa^b = 0$.
- Method 3. Utilizing a formula,
  - $-\epsilon^{acd}\epsilon^{bef}h_{ce}h_{df} = \left(h_+^2 + h_\times^2\right)\kappa^a\kappa^b$
  - Find an eigenvector of the above matrix with non-zero eigenvalue.

The Inverse Problem

- Given
  - $h_1, \cdots, h_N$: N independent channels
  - $t_1, \cdots t_M$: M sampling time
  - $h_i\left(t_k\right)$: GW signal for i-th channel at t_k
  - total number: NM
- How to determine h_{ab}?
- Unknowns
  - $h_+\left(t_k\right)$, $h_\times\left(t_k\right)$, θ, ϕ
  - total number: $2M + 2$
- What is the minimum number of independent channels required to determine h_{ab}?
- Answer: $N \geq 3$

Inverse Problem Without Noises

Case I: 5 Channels

- $\mathscr{S}$: Vector space for symmetric traceless $(0,2)$ tensors
- Projection operator for $\mathscr{S}$
  - $\Gamma^{ab}{}_{cd} = \gamma^{a}{}_{(c}\gamma^{b}{}_{d)} - \frac{1}{3}\gamma^{ab}\gamma_{cd}$
  - where
  - $\gamma^{a}{}_{b} = \delta^{a}{}_{b} + n^{a}n_{b}$: spatial metric
  - $n^{a} = -g^{ab}\left(dt\right)_{b}$: normal vector to the hypersurface of $t = \mathrm{const}$
- The set of independent detector tensors $\left\{D_i^{ab} : i = 1,\cdots,5\right\}$ forms basis for $\mathscr{S}$.
- Given the basis $\left\{D_i^{ab}\right\}$, we can uniquely determine dual basis $\left\{D^i_{ab}\right\}$ such that
  - $D^i_{ab}D_j^{ab} = \delta^{i}{}_{j}$
  - $D_i^{ab}D^i_{cd} = \Gamma^{ab}{}_{cd}$
- Then
  - $h_{ab} = D^i_{ab}h_i$

Case I: 5 Channels

- Algorithm to determine the dual basis $\left\{D^i_{ab}\right\}$
  - Step 1. $\Gamma_{ij} \equiv \Gamma_{abcd} D^{ab}_i D^{cd}_j$
  - Step 2. Get the inverse Γ^{ij} such that $\Gamma^{ik}\Gamma_{kj} = \delta^i{}_j$
  - Step 3. $D^i_{ab} = \Gamma^{ij}\Gamma_{abcd} D^{cd}_j$
- Check the properties
  - $D^i_{ab} D^{ab}_j = \Gamma^{ik}\Gamma_{abcd} D^{cd}_k D^{ab}_j = \Gamma^{ik}\Gamma_{jk} = \delta^i{}_j$
  - $D^{ab}_i D^i_{cd} = D^{ab}_i \Gamma^{ij}\Gamma_{cdef} D^{ef}_j = \Gamma^{abef}\Gamma_{cdef} = \Gamma^{ab}{}_{cd}$

Example: SOGRO without zz channel

- For the terrestrial SOGRO, the channel for h_{zz} component is not available due to the gravity bias.
- Detector tensors
  - $D_1^{ab} = \Gamma^{ab}{}_{cd}\hat{x}^c\hat{x}^d = \frac{2}{3}\hat{x}^a\hat{x}^b - \frac{1}{3}\hat{y}^a\hat{y}^b - \frac{1}{3}\hat{z}^a\hat{z}^b$
  - $D_2^{ab} = \Gamma^{ab}{}_{cd}\hat{y}^c\hat{y}^d = -\frac{1}{3}\hat{x}^a\hat{x}^b + \frac{2}{3}\hat{y}^a\hat{y}^b - \frac{1}{3}\hat{z}^a\hat{z}^b$
  - $D_3^{ab} = \frac{1}{2}\left(\hat{x}^a\hat{y}^b + \hat{y}^a\hat{x}^b\right)$ $\quad D_4^{ab} = \frac{1}{2}\left(\hat{y}^a\hat{z}^b + \hat{z}^a\hat{y}^b\right)$
  - $D_5^{ab} = \frac{1}{2}\left(\hat{z}^a\hat{x}^b + \hat{x}^a\hat{z}^b\right)$
- Gamma matrix
  - $\Gamma_{ij} = \begin{bmatrix} 2/3 & -1/3 & 0 & 0 & 0 \\ -1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}$
  - $\Gamma^{ij} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$
- Dual basis
  - $D^1_{ab} = \hat{x}_a\hat{x}_a - \hat{z}_a\hat{z}_b$ $\quad D^2_{ab} = \hat{y}_a\hat{y}_a - \hat{z}_a\hat{z}_b$
  - $D^3_{ab} = \hat{x}_a\hat{y}_a + \hat{y}_a\hat{x}_b$ $\quad D^4_{ab} = \hat{y}_a\hat{z}_a + \hat{z}_a\hat{y}_b$
  - $D^5_{ab} = \hat{z}_a\hat{x}_a + \hat{x}_a\hat{z}_b$

Case II: 4 Channels

- Find the fifth detector tensor D_5^{ab} such that $\left\{D_i^{ab} : i = 1,\cdots,5\right\}$ forms a basis for $\mathcal{S}$.
  - Step 1. Initial guess for D_5^{ab}
  - Step 2. For $i = 1,\cdots,4$
    - $D_5^{ab} \leftarrow D_5^{ab} - D_i^{ab}\left\langle D_5, D_i\right\rangle / \left\langle D_i, D_i\right\rangle$ where $\langle A, B\rangle \equiv \Gamma_{abcd}A^{ab}B^{cd}$
  - Step 3. If $D_5^{ab} = 0$ go to Step 1 again.
- Find h_5 by solving $\det\left(h_{ij}\right) = 0$
  - It is because h_{ab} has a zero eigenvalue from $h_{ab}\kappa^b = 0$.
  - $0 = 3!\left(\epsilon^{123}\right)^2 \det\left(h_{ij}\right) = \epsilon^{ace}\epsilon^{bdf}h_{ab}h_{cd}h_{ef} = \left(\epsilon^{ace}\epsilon^{bdf}D^i_{ab}D^j_{cd}D^k_{ef}\right)h_ih_jh_k$
- SOGRO example
  - $0 = -h_1h_5^2 + 2h_3h_4h_5 - h_1^2h_2 - h_1h_2^2 + h_1h_3^2 + h_2h_3^2 - h_2h_4^2$

Case III: 3 Channels

- Given
  - D_i^{ab} for $i = 1,2,3$
  - $h_i(t)$, $h_i(t')$: GW signal at two different times
- Unknowns
  - θ, ϕ, $h_+(t)$, $h_\times(t)$, $h_+(t')$, $h_\times(t')$
- Solve
  - $h_1(t) = F_1^+(\theta,\phi)\, h_+(t) + F_1^\times(\theta,\phi)\, h_\times(t)$
  - $h_2(t) = F_2^+(\theta,\phi)\, h_+(t) + F_2^\times(\theta,\phi)\, h_\times(t)$
  - $h_3(t) = F_3^+(\theta,\phi)\, h_+(t) + F_3^\times(\theta,\phi)\, h_\times(t)$
  - $h_1(t') = F_1^+(\theta,\phi)\, h_+(t') + F_1^\times(\theta,\phi)\, h_\times(t')$
  - $h_2(t') = F_2^+(\theta,\phi)\, h_+(t') + F_2^\times(\theta,\phi)\, h_\times(t')$
  - $h_3(t') = F_3^+(\theta,\phi)\, h_+(t') + F_3^\times(\theta,\phi)\, h_\times(t')$
  - where
  - $F_i^A(\theta,\phi) \equiv D_i^{ab} e_{ab}^A(\theta,\phi)$ for $A = +, \times$
- SOGRO example
  - Can we construct h_{ab} with only xx, yy, xy channels? Is a two dimensional SOGRO possible?
  - We have to investigate degeneracy of solutions.

Inverse Problem with Noises

Channel Outputs and Noises

- Channel Outputs
  - $s_i(t) = h_i(t) + n_i(t)$
- Gaussian stationary Noises
  - $\left\langle n_i(t) \right\rangle = 0$
  - $\left\langle n_i(t)\, n_j(t+\tau) \right\rangle = R^n_{ij}(\tau)$
- Channel output correlation
  - $\left\langle \tilde{s}^*_i(\omega)\, \tilde{s}_j(\omega) \right\rangle = \tilde{h}^*_i(\omega)\, \tilde{h}_j(\omega) + S_{ij}(\omega)$
  - where
  - $S_{ij}(\omega) \equiv \left\langle \tilde{n}^*_i(\omega)\, \tilde{n}_j(\omega) \right\rangle$ is Hermitian matrix.
- If each channels are independent, we can reasonably assume that S_{ij} is positive-definite. Then, we can introduce a unitary matrix U which diagonalize S_{ij}.
  - $\left\langle \tilde{s}'^*_i(\omega)\, \tilde{s}'_j(\omega) \right\rangle = \tilde{h}'^*_i(\omega)\, \tilde{h}'_j(\omega) + S'_{ij}(\omega)$
  - where $S'_{ij} \equiv S_{kl} U^k{}_i U^l{}_j$ is diagonal
  - $\tilde{s}'_i \equiv \tilde{s}_j U^j{}_i$ $\qquad$ $\tilde{h}'_i \equiv \tilde{h}_j U^j{}_i$

The Inverse Problem

- Unknowns
  - $\tilde{h}_{ab}\left(\omega,\theta,\phi\right) = \tilde{h}_A(\omega)\, e^A_{ab}\left(\theta,\phi\right)$
  - θ, ϕ, $\tilde{h}_+(\omega)$, $\tilde{h}_\times(\omega)$
  - total number: 6 = 1 + 1+ 2 + 2
- We utilize off-diagonal part of the correlation $\left\langle \tilde{s}_i'^*(\omega)\,\tilde{s}_j'(\omega)\right\rangle$ for $i \neq j$.
- Minimum number of channels to determine $\tilde{h}_{ab}$ is 3:
  - $\left\langle \tilde{s}_1'^*(\omega)\,\tilde{s}_2'(\omega)\right\rangle = \tilde{h}_1'^*(\omega)\,\tilde{h}_2'(\omega) = U^i{}_1 U^j{}_2 F^A_i\left(\theta,\phi\right) F^B_j\left(\theta,\phi\right) \tilde{h}^*_A(\omega)\,\tilde{h}_B(\omega)$
  - $\left\langle \tilde{s}_2'^*(\omega)\,\tilde{s}_3'(\omega)\right\rangle = \tilde{h}_2'^*(\omega)\,\tilde{h}_3'(\omega) = U^i{}_2 U^j{}_3 F^A_i\left(\theta,\phi\right) F^B_j\left(\theta,\phi\right) \tilde{h}^*_A(\omega)\,\tilde{h}_B(\omega)$
  - $\left\langle \tilde{s}_3'^*(\omega)\,\tilde{s}_1'(\omega)\right\rangle = \tilde{h}_3'^*(\omega)\,\tilde{h}_1'(\omega) = U^i{}_3 U^j{}_1 F^A_i\left(\theta,\phi\right) F^B_j\left(\theta,\phi\right) \tilde{h}^*_A(\omega)\,\tilde{h}_B(\omega)$

Summary and Discussion

- To localization the source direction in multi-channel detectors, we need to construct the metric perturbation tensor, which is so-called the inverse problem.
- In the absence of noise, this reduces to solving a set of linear or non-linear equations.
- When noise is present, we can make use of the off-diagonal components of the noise correlation matrix.
- Even in this chase, additional errors may arise due to the finite number of sampling points.
- To address this, we can apply methods for solving overdetermined system. For example, by minimizing $S'^{ij}\left(\left\langle \tilde{s}_i'^{*}\tilde{s}_j \right\rangle - \tilde{h}_i'^{*}\tilde{h}_j\right)$.
- A practical demonstration of this method will be part of our future work.
- Thank you for listening 😁